If A is the adjacency matrix of a graph (multigraph, could have loops, could be directed),

then $(A^n)_{ij}$ = # of paths from vertex i to vertex j of length n.

$(A^n)_{22}$ = # of circuits of length n starting & ending at vertex 2.

Total # of circuits in the graph = ?

Guess: $(A^n)_{11} + (A^n)_{22} + \cdots + (A^n)_{kk}$

↑ last vertex.

sort of correct, but you are count each circuit more than once (once for each possible starting vertex.)

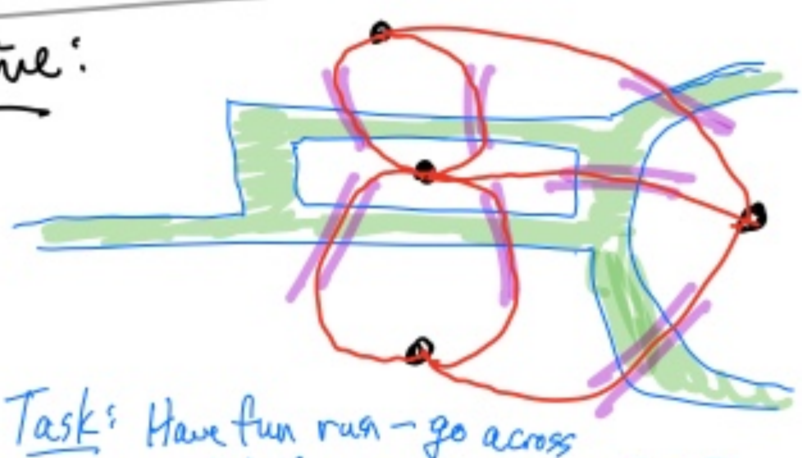
Special case: If the multigraph has <u>no loops</u>, then

# of circuits of length 3 is

$\frac{1}{3}\left[(A^3)_{11} + (A^3)_{22} + \cdots\right]$

↑ every circuit has 3 different vertices

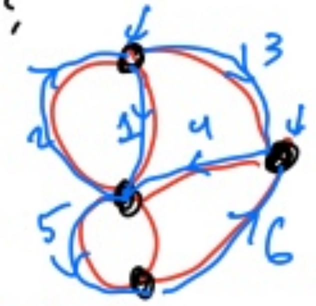
Last time:



Task: Have fun run – go across each bridge once. How do we do it?

Converted to a graph theory question:

Can we find a path that traverses every edge exactly once?



One thought: it's impossible.

We can almost solve it using the blue path, but one edge is left out.

Definitions Given a connected graph G, an Euler path is a path that traverses every edge exactly once.
An Euler circuit is a circuit where each edge of G is traversed exactly once.

Big Theorem Euler's Graph Theorem — A connected undirected graph (multigraph, with loops allowed) contains an Euler path if there are 0 or 2 vertices of odd degree.
It contains an Euler circuit if there are no vertices of odd degree.

Proof. Suppose there exists an Euler path from vertex a to vertex b. Every time it approaches a vertex & leaves the vertex to go somewhere else, it uses two edges that touch the vertex. If the Euler path is

$a \to v_2 \to v_3 \to \cdots \to v_{k-1} \to b$, then all of the vertices not equal to a or b must have even degree, because we go in & out through those vertices.

If $a \neq b$, then both a & b must be odd. (the first edge leaves a, using only 1 edge. the last edge enters b, using only one edge).

If $a = b$ (circuit case) all vertices must be even.

<u>Converse</u>: Suppose a graph has no vertices of odd degree:

Sketch

① try to make an Euler circuit until you get stuck.

② One of the vertices has ≥ 2 edges coming out of it. Start there, run through the circuit, then keep going.

⋮ prove that eventually you make the circuit over the whole graph.

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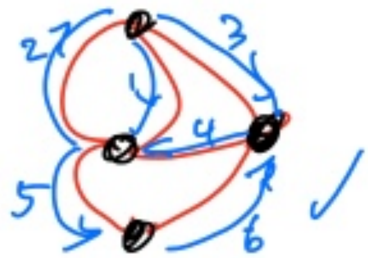
Suppose the graph has 2 vertices of odd degree. Then add one extra edge connecting those two vertices. Then you have a graph with no odd vertices. $\longrightarrow$ Make an Euler circuit, delete the extra edge to make the path. ∎

Example: ①

degrees of vertices: 3, 3, 3, 5

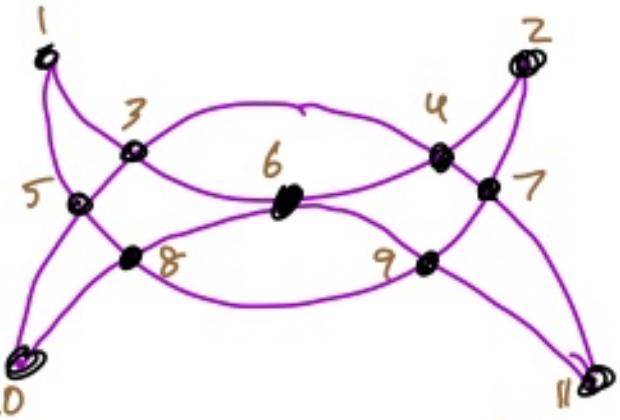
$\Rightarrow$ impossible.

(2)



Degrees:
3, 3, 2, 4
⇒ possible

(3)

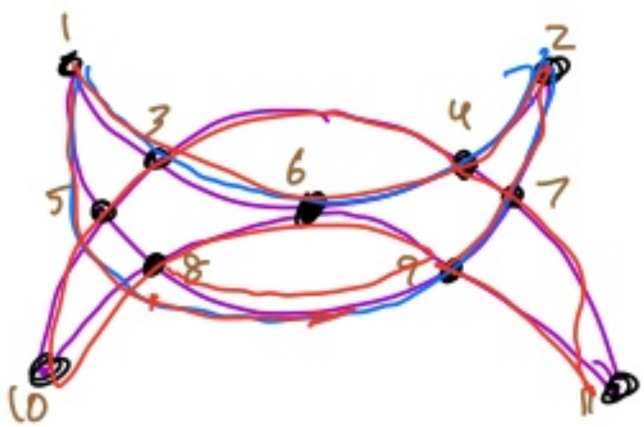


Can you trace this picture without lifting the pencil?

Degrees.

vertex:	1	2	3	4	5	6	7	8	9	10	11
degree	2	2	4	4	4	4	4	4	4	2	2

There is an Euler circuit!!



Quiz

1. Favorite Chips
2. Define <u>degree</u> of a vertex v in a graph G.
3. A graph is called <u>connected</u> if .....
   (finish definition)
4. In the adjacency matrix A of a graph,
   $A_{ij} = \#$ of ___ from ___ to ___.